Reaction-rate formula in out-of-equilibrium quantum-field theory

A. Niégawa,* K. Okano,† and H. Ozaki‡
Department of Physics, Osaka City University, Sumiyoshi-ku, Osaka 558-8585, Japan
(Received 16 August 1999; published 8 February 2000)

A complete derivation, from first principles, of the reaction-rate formula for a generic reaction taking place in an out-of-equilibrium quantum-field system is given. It is shown that the formula involves no finite-volume correction. Each term of the reaction-rate formula represents a set of physical processes that contributes to the reaction under consideration.

PACS number(s): 11.10.Wx, 12.38.Bx, 12.38.Mh

I. INTRODUCTION

Ultrarelativistic heavy-ion-collision experiments at the BNL Relativistic Heavy Ion Collider (RHIC) and at the CERN Large Hadron Collider (LHC) will soon start in anticipation of producing a quark-gluon plasma (QGP). Confirmation of QGP formation is realized through analyzing rates of various reactions taking place in a QGP. So far, the reaction-rate formula is derived for reactions taking place in the system in thermal and chemical equilibrium [1–4]. The actual QGP is, however, not in equilibrium but is an expanding nonequilibrium system.

In this paper, as a generalization of [1–4], we present a first-principles derivation of the reaction-probability formula for reactions occurring in a nonequilibrium system. We find that the formula involves no finite-volume corrections. We also find from the procedure of derivation that different contributions to the reaction-probability formula have a clear physical interpretation, which is summarized as ‘‘out-of-equilibrium cutting rules.’’

In Sec. II, we derive from first principles the formula for the transition probability of a generic reaction taking place in a nonequilibrium system. In Sec. III, specializing to quasi-uniform systems near equilibrium or nonequilibrium quasi-stationary systems,[1] we further deduce the formula, finding that the formula is written in terms of the closed-time-path formalism of real-time thermal field theory [5]. In Sec. IV, we present a calculational procedure of a generic reaction-probability formula obtained in Sec. III.

II. NONEQUILIBRIUM REACTION-PROBABILITY FORMULA

A. Preliminaries

The formalism presented in this paper can be applied to a broad class of theories including QCD (cf. the end of Sec. III), but, for simplicity of presentation, we take a system of self-interacting, neutral scalars ϕ's with mass m and $\lambda\phi^4$ interaction. The system is inside a cube with volume $V=L^3$. Employing periodic boundary conditions, we label the single-particle basis by its momentum $\mathbf{p_k}=2\pi\mathbf{k}/L$, $k_j=0,\pm1,\pm2,\ldots,\pm\infty$ $(j=1,2,3)$.

Physically interesting reactions are of the following generic type:

$$\{A\}+\text{nonequilibrium system}\rightarrow\{B\}+\text{anything}. \tag{2.1}$$

Here $\{A\}$ and $\{B\}$ designate groups of particles, which are different from ϕ. Examples are highly virtual particles, heavy particles, and particles interacting weakly with ϕ's. The generalization to a more general process, where among $\{A\}$ and/or $\{B\}$ are ϕ's, is straightforward (cf. [4]). For definiteness, let us assume that $\{A\}$ consists of $l\,\Phi$'s and $\{B\}$ consists of $l'\,\Phi$'s. Here Φ is a heavy neutral scalar of mass M, so that Φ is absent in the system. For simplicity of presentation, we assume a Φ-ϕ coupling to be of the form $-g\Phi\phi^n/n!$ $(n\geqslant2)$.

The transition or reaction probability $\mathcal{P}$ of the process (2.1) is written as

$$\mathcal{P}=\mathcal{N}/\mathcal{D}, \tag{2.2a}$$

$$\mathcal{N}\equiv\sum_{\{\mathbf{k}\}}\sum_{\{n_\mathbf{k}\}}\sum_{\{m_\mathbf{k}\}}\sum_{\{n'_\mathbf{k}\}}\langle\{A\};\{m_\mathbf{k}\}|S^\dagger|\{n'_\mathbf{k}\};\{B\}\rangle \times\langle\{B\};\{n'_\mathbf{k}\}|S|\{n_\mathbf{k}\};\{A\}\rangle\langle\{n_\mathbf{k}\}|\rho|\{m_\mathbf{k}\}\rangle\mathbf{S}, \tag{2.2b}$$

$$\mathcal{D}\equiv\sum_{\{\mathbf{k}\}}\sum_{\{n_\mathbf{k}\}}\sum_{\{m_\mathbf{k}\}}\sum_{\{n'_\mathbf{k}\}}\langle\{m_\mathbf{k}\}|S^\dagger|\{n'_\mathbf{k}\}\rangle\langle\{n'_\mathbf{k}\}|S|\{n_\mathbf{k}\}\rangle \times\langle\{n_\mathbf{k}\}|\rho|\{m_\mathbf{k}\}\rangle\mathbf{S}. \tag{2.2c}$$

Here $\mathbf{S}$ is the symmetry factor [6], ρ is the density matrix, and $\langle\{B\};\{n'_\mathbf{k}\}|S|\{n_\mathbf{k}\};\{A\}\rangle$ is an S-matrix element of the *vacuum-theory process*,

$$\{A\}+\{n_\mathbf{k}\}\rightarrow\{B\}+\{n'_\mathbf{k}\},$$

where $\{n_\mathbf{k}\}$ denotes the group of ϕ's, which consists of the number $n_\mathbf{k}$ of $\phi_\mathbf{k}$ (ϕ in a mode $\mathbf{k}$). In Eqs. (2.2), $\Sigma_{\{\mathbf{k}\}}$ denotes summation over momentum/momenta of ϕ/ϕ's in the final state $|\{n'_\mathbf{k}\};\{B\}\rangle$, and of ϕ/ϕ's in the ‘‘two’’ initial states $|\{m_\mathbf{k}\};\{A\}\rangle$ and $|\{n_\mathbf{k}\};\{A\}\rangle$. (Among the final states

*Electronic address: niegawa@sci.osaka-cu.ac.jp
†Electronic address: okano@sci.osaka-cu.ac.jp
‡Electronic address: hozaki@sci.osaka-cu.ac.jp

[1]The framework for dealing with such systems is comprehensively discussed in [5].

$|\{n'_{\mathbf{k}}\};\{B\}\rangle$, is $|0;\{B\}\rangle$. This is also the case for ''two'' initial states.) Note that the perturbation series for $\mathcal{D}$ starts from 1,

$$\mathcal{D}=1+\cdots.$$

It is to be noted that $\{A\}$ and $\{B\}$ in $\langle S\rangle$, which we write $\{A,B\}_S$, are not necessarily involved in one connected part of $\langle S\rangle$. This is also the case for $\{A,B\}_{S^\dagger}$. We assume that, in $W\equiv\langle S^\dagger\rangle\langle S\rangle$, $\{A,B\}_S$, and $\{A,B\}_{S^\dagger}$ are involved in one connected part $W_c(\in W)$. Then, W consists, in general, of W_c and other parts which are disconnected with W_c and include only ϕ's. The generalization to other cases is straightforward [4]. The form of ρ in Eqs. (2.2) should be remarked on. Let us recall the following two facts. On the one hand, the statistical ensemble is defined by the density matrix at the very initial time $t_i(\sim-\infty)$. On the other hand, in constructing a perturbative framework, an adiabatic switching off of the interaction is required [7,3]. Then, ρ in Eqs. (2.2) is a functional of the in-field $\phi_{in}\ (t_i,\mathbf{x})$ that constitutes the basis of perturbation theory.

As will be seen below, diagrammatic analysis shows that $\mathcal{N}$, Eq. (2.2b), takes the form

$$\mathcal{N}=\mathcal{N}_{\rm con}\mathcal{D}, \tag{2.3}$$

where $\mathcal{N}_{\rm con}$ corresponds to a connected diagram and $\mathcal{D}$ is as in Eq. (2.2c). Then, we have

$$\mathcal{P}=\mathcal{N}_{\rm con}.$$

The S-matrix element in vacuum theory is obtained through an application of the reduction formula [2,4]:

$$\langle\{B\};\{n'_{\mathbf{k}}\}|S|\{n_{\mathbf{k}}\};\{A\}\rangle=\prod_{j=1}^{l}\ (iK_{j,\Phi_j})\prod_{m=1}^{l'}\ (iK^*_{m,\Phi_m})\langle 0|\prod_{\mathbf{k}}\ T\left[\left\{\sum_{i_{\mathbf{k}}=0}^{n_{\mathbf{k}}}\ \sum_{i'_{\mathbf{k}}=0}^{n'_{\mathbf{k}}}\ \delta(n_{\mathbf{k}}-i_{\mathbf{k}};\quad n'_{\mathbf{k}}-i'_{\mathbf{k}})N^{n_{\mathbf{k}}n'_{\mathbf{k}}}_{i_{\mathbf{k}}i'_{\mathbf{k}}}\right.\right.$$
$$\left.\left.\times\prod_{n'=1}^{i'_{\mathbf{k}}}\ (iK^*_{\mathbf{k},n'})\prod_{n=1}^{i_{\mathbf{k}}}\ (iK_{\mathbf{k},n})\prod_{n'=1}^{i'_{\mathbf{k}}}\ \phi_{n'}\prod_{n=1}^{i_{\mathbf{k}}}\ \phi_n\right\}\prod_{j=1}^{l}\ \Phi_j\prod_{m=1}^{l'}\ \Phi_m\right]|0\rangle, \tag{2.4}$$

where T is the time-ordering symbol and

$$N^{n_{\mathbf{k}}n'_{\mathbf{k}}}_{i_{\mathbf{k}}i'_{\mathbf{k}}}\equiv\left\{\binom{n'_{\mathbf{k}}}{i'_{\mathbf{k}}}\binom{n_{\mathbf{k}}}{i_{\mathbf{k}}}\frac{1}{i'_{\mathbf{k}}!\,i_{\mathbf{k}}!}\right\}^{1/2}. \tag{2.5}$$

In Eq. (2.4), $\delta(\cdots;\cdots)$ denotes the Kronecker's δ symbol and

$$K_{\mathbf{k},n}\cdots\phi_n\equiv\frac{1}{\sqrt{Z_\phi}}\int\ d^4x\, f_{\mathbf{p_k}}(x)(\Box+m^2)\cdots\phi(x),$$

$$K_{j,\Phi_j}\cdots\Phi_j\equiv\frac{1}{\sqrt{Z_\Phi}}\int\ d^4x\, F_j(x)(\Box+M^2)\cdots\Phi(x),$$

$$K^*_{m,\Phi_m}\cdots\Phi_m\equiv\frac{1}{\sqrt{Z_\Phi}}\int\ d^4x\, G^*_m(x)(\Box+M^2)\cdots\Phi(x). \tag{2.6}$$

Here

$$f_{\mathbf{p_k}}(x)=\frac{1}{\sqrt{2E_{\mathbf{k}}V}}e^{-iP_{\mathbf{k}}\cdot x},\quad (E_{\mathbf{k}}=\sqrt{p_{\mathbf{k}}^2+m^2}),$$

with $P^\mu_{\mathbf{k}}\equiv(E_{\mathbf{k}},\mathbf{p_k})$ and $F_j(x)[G^*_m(x)]$ the wave function of the jth $\Phi\ (\in\{A\})\ [m\text{th}\ \Phi\ (\in\{B\})]$. Z's in Eq. (2.6) are the wave-function renormalization constants. It is to be noted that, in Eq. (2.4), among $n_{\mathbf{k}}\ (n'_{\mathbf{k}})$ of $\phi_{\mathbf{k}}$'s in the initial (final) state, $i_{\mathbf{k}}\ (i'_{\mathbf{k}})$ of $\phi_{\mathbf{k}}$'s are absorbed in (emitted from) the $i_{\mathbf{k}}\ (i'_{\mathbf{k}})$ vertices in $\langle S\rangle$. The remaining $n_{\mathbf{k}}-i_{\mathbf{k}}\ (=n'_{\mathbf{k}}-i'_{\mathbf{k}})$ of $\phi_{\mathbf{k}}$'s are merely spectators, which reflects only on the statistical factor in $\mathcal{F}_i$ in Eq. (3.19) below.

$\langle S\rangle$ in $\mathcal{D}$ in Eq. (2.2c) is given by a similar expression to Eq. (2.4), where factors related to the Φ fields are deleted.

From the form for $\langle S\rangle$, Eq. (2.4), we see that the permutation of $\phi_n\ (n=1,\dots,i_{\mathbf{k}})$ and the permutation of $\phi_{n'}\ (n'=1,\dots,i'_{\mathbf{k}})$ give the same Feynman diagram (in vacuum theory), and then $i_{\mathbf{k}}!\,i'_{\mathbf{k}}!$ same diagrams emerge. Taking this fact into account, we may write Eq. (2.4) in the form

$$\langle\{B\};\{n'_{\mathbf{k}}\}|S|\{n_{\mathbf{k}}\};\{A\}\rangle=\left(\prod_{j=1}^{l}\ \int\ d^4x_jF_j(x_j)\right)\left(\prod_{m=1}^{l'}\ \int\ d^4y_mG^*_m(y_m)\right)\sum_{\{i_{\mathbf{k}}\}}\left[\prod_{\mathbf{k}}\ N^{n_{\mathbf{k}}n'_{\mathbf{k}}}_{i_{\mathbf{k}}i'_{\mathbf{k}}}i_{\mathbf{k}}!\,i'_{\mathbf{k}}!\left(\prod_{j=1}^{i_{\mathbf{k}}}\ \int\ d^4\xi_{\mathbf{k}j}f_{\mathbf{p_k}}(\xi_{\mathbf{k}j})\right)\right.$$
$$\left.\times\left(\prod_{j=1}^{i'_{\mathbf{k}}}\ \int\ d^4\zeta_{\mathbf{k}j}f^*_{\mathbf{p_k}}(\zeta_{\mathbf{k}j})\right)\right]\mathcal{A}(\{y\},\{\zeta\};\{\xi\},\{x\}), \tag{2.7}$$

where $i'_{\mathbf{k}}=n'_{\mathbf{k}}-n_{\mathbf{k}}+i_{\mathbf{k}}$ and $\mathcal{A}$ is the truncated Green function in configuration space (in vacuum theory), and, e.g., $\{y\}$ collectively denotes $y_1,y_2,\dots,y_{l'}$.

Among the Feynman diagrams for $\mathcal{A}$ are some diagrams in which some ξ's ($\in\{\xi\}$) [ζ's ($\in\{\zeta\}$)] coincide with x's ($\in\{x\}$) and/or y's ($\in\{y\}$) and/or ζ's ($\in\{\zeta\}$)[ξ's ($\in\{\xi\}$)]. In such cases, $\mathcal{A}$ is understood to include the corresponding δ functions, e.g., $\delta^4(\xi_{\mathbf{k}j}-x_i)$.

The expression for $\langle S^\dagger\rangle$, the complex conjugate of $\langle S\rangle$, is obtained by taking the complex conjugate of Eq. (2.4) or Eq. (2.7), where we make the substitution [cf. Eqs. (2.2b) and (2.2c)],

$$n_{\mathbf{k}}\to m_{\mathbf{k}},\quad n'_{\mathbf{k}}\to m'_{\mathbf{k}}(=n'_{\mathbf{k}}),\quad i_{\mathbf{k}}\to j_{\mathbf{k}},\quad i'_{\mathbf{k}}\to j'_{\mathbf{k}}.$$

This applies also to the expression for $\langle S^\dagger\rangle$ in Eq. (2.2c).

Substitution of $W=\langle S^\dagger\rangle\langle S\rangle$ into Eq. (2.2b) yields, with obvious notation,

$$\begin{aligned}\mathcal{N}=&\left(\prod_{j=1}^{l}\int d^4x_j d^4x'_j F_j(x_j)F_j^*(x'_j)\right)\\&\times\left(\prod_{m=1}^{l'}\int d^4y_m d^4y'_m G_m^*(y_m)G_m(y'_m)\right)\\&\times\sum_{\{\mathbf{k}\}}\sum_{\{i_{\mathbf{k}}\}}\sum_{\{j_{\mathbf{k}}\}}\sum_{\{i'_{\mathbf{k}}\}}\sum_{\{j'_{\mathbf{k}}\}}\left[\prod_{\mathbf{k}}\left(\prod_{j=1}^{i_{\mathbf{k}}}\int d^4\xi_{\mathbf{k}j}f_{\mathbf{p_k}}(\xi_{\mathbf{k}j})\right)\right.\\&\times\left(\prod_{j=1}^{i'_{\mathbf{k}}}\int d^4\zeta_{\mathbf{k}j}f^*_{\mathbf{p_k}}(\zeta_{\mathbf{k}j})\right)\left(\prod_{j=1}^{j_{\mathbf{k}}}\int d^4\xi'_{\mathbf{k}j}f^*_{\mathbf{p_k}}(\xi'_{\mathbf{k}j})\right)\\&\left.\times\left(\prod_{j=1}^{j'_{\mathbf{k}}}\int d^4\zeta'_{\mathbf{k}j}f_{\mathbf{p_k}}(\zeta'_{\mathbf{k}j})\right)\right]\\&\times\mathcal{S}\mathcal{W}(\{x'\},\{\xi'\};\{\zeta'\},\{y'\}:\{y\},\{\zeta\};\{\xi\},\{x\})\mathbf{S}.\end{aligned}\tag{2.8}$$

Here $i'_{\mathbf{k}}=n'_{\mathbf{k}}-n_{\mathbf{k}}+i_{\mathbf{k}}$, $j'_{\mathbf{k}}=n'_{\mathbf{k}}-m_{\mathbf{k}}+j_{\mathbf{k}}$, $\mathcal{W}=\mathcal{A}^*\mathcal{A}$, and

$$\mathcal{S}\equiv\sum_{\{n_{\mathbf{k}}\}}\left(\prod_{\mathbf{k}}N^{m_{\mathbf{k}}n'_{\mathbf{k}}}_{j_{\mathbf{k}}j'_{\mathbf{k}}}N^{n_{\mathbf{k}}n'_{\mathbf{k}}}_{i_{\mathbf{k}}i'_{\mathbf{k}}}i_{\mathbf{k}}!\,i'_{\mathbf{k}}!\,j_{\mathbf{k}}!\,j'_{\mathbf{k}}!\right)\langle\{n_{\mathbf{k}}\}|\rho|\{m_{\mathbf{k}}\}\rangle.\tag{2.9}$$

B. Statistical factor $\mathcal{S}$

Here, it is convenient to introduce creation and annihilation operators $a^\dagger_{\mathbf{p_k}}$ and $a_{\mathbf{p_k}}$, which satisfy $[a_{\mathbf{p_k}},a^\dagger_{\mathbf{p_{k'}}}]=\delta_{\mathbf{k},\mathbf{k}'}$, and $[a_{\mathbf{p_k}},a_{\mathbf{p_{k'}}}]=0$. A Fock space $\mathcal{F}$ is constructed on $|0\rangle$, which is defined by $a_{\mathbf{p_k}}|0\rangle=0$. For the vector $|\rangle$ ($\in\mathcal{F}$) that satisfies $a^\dagger_{\mathbf{p_k}}a_{\mathbf{p_k}}|\rangle=n_{\mathbf{p_k}}|\rangle(n_{\mathbf{p_k}}=0,1,2,\dots)$, we use the same notation as in Eq. (2.9), $|\{n_{\mathbf{k}}\}\rangle$, since no confusion arises. A key observation here is that, using the form (2.5), one can easily show that $\mathcal{S}$, Eq. (2.9), may be represented as

$$\begin{aligned}\mathcal{S}=&\sum_{\{n_{\mathbf{k}}\}}\langle\{m_{\mathbf{k}}\}|\left(\prod_{l=1}^{j}a^\dagger_{\mathbf{p}'_l}\right)\left(\prod_{l=1}^{j'}a_{\mathbf{q}'_l}\right)\left(\prod_{l=1}^{i'}a^\dagger_{\mathbf{q}_l}\right)\\&\times\left(\prod_{l=1}^{i}a_{\mathbf{p}_l}\right)|\{n_{\mathbf{k}}\}\rangle\langle\{n_{\mathbf{k}}\}|\rho|\{m_{\mathbf{k}}\}\rangle\\&\equiv\left\langle\left(\prod_{l=1}^{j}a^\dagger_{\mathbf{p}'_l}\right)\left(\prod_{l=1}^{j'}a_{\mathbf{q}'_l}\right)\left(\prod_{l=1}^{i'}a^\dagger_{\mathbf{q}_l}\right)\left(\prod_{l=1}^{i}a_{\mathbf{p}_l}\right)\right\rangle,\end{aligned}\tag{2.10}$$

where we write

$$\{\mathbf{p}_1,\cdots,\mathbf{p}_i\}=\{\cdots,\underbrace{\mathbf{p_k},\cdots,\mathbf{p_k}}_{i_{\mathbf{k}}},\cdots\},$$

and then $i=\Sigma_{\mathbf{k}}i_{\mathbf{k}}$. Similarly, $i'=\Sigma_{\mathbf{k}}i'_{\mathbf{k}}$, $j=\Sigma_{\mathbf{k}}j_{\mathbf{k}}$, and $j'=\Sigma_{\mathbf{k}}j'_{\mathbf{k}}$. Note that $\langle\{n_{\mathbf{k}}\}|$ and $|\{m_{\mathbf{k}}\}\rangle$, in between which ρ is sandwiched, are as in Eqs. (2.2) and (2.9).

Let us write $\mathcal{S}$, for short, as $\mathcal{S}=\langle b_1b_2\dots b_N\rangle$ $(N=i+j+i'+j')$. Let $l_1,\dots,l_m$ be a solution in positive integers of

$$\sum_{j=1}^{m}l_j=N\quad(1\leq m\leq N).\tag{2.11}$$

Pick out l_1 b's out of $b_1,b_2,\dots,b_N$ and pick out l_2 b's out of remaining b's, and so on, to make m groups,

$$\{b_1\cdots b_{i_{l_1}}\}\{b_{i_{l_1+1}}\cdots b_{i_{l_1+l_2}}\}\cdots\{b_{i_{N-l_m+1}}\cdots b_{i_N}\},\tag{2.12}$$

where $1<i_{l_1+1}<i_{l_1+l_2+1}<\cdots<i_{N-l_m+1}\leq N$. In Eq. (2.12), let b_l and $b_{l'}$ be in between one set of curly brackets. Then, if $l<l'$, b_l is located at the left of $b_{l'}$ and vice versa. We are now in a position to write

$$\begin{aligned}\mathcal{S}(b_1\cdots b_N)=&\sum_{m=1}^{N}\sum_{l'\text{s}}\sum_{\text{gr}}\mathcal{S}_c(b_1\cdots b_{i_{l_1}})\\&\times\mathcal{S}_c(b_{i_{l_1+1}}\cdots b_{i_{l_1+l_2}})\cdots\mathcal{S}_c(b_{i_{N-l_m+1}}\cdots b_{i_N}).\end{aligned}\tag{2.13}$$

Here, the second summation $\Sigma_{l'\text{s}}$ runs over all solutions in integers of Eq. (2.11) and the third summation Σ_{gr} runs over all ways of making m groups as in Eq. (2.12). From Eq. (2.13), $\mathcal{S}_c$ is determined iteratively. For example,

$$\mathcal{S}_c(b_1b_2)=\mathcal{S}(b_1b_2)-\mathcal{S}(b_1)\mathcal{S}(b_2),$$

$$\begin{aligned}\mathcal{S}_c(b_1b_2b_3)=&\mathcal{S}(b_1b_2b_3)-\mathcal{S}_c(b_1b_2)\mathcal{S}(b_3)\\&-\mathcal{S}_c(b_1b_3)\mathcal{S}(b_2)-\mathcal{S}(b_1)\mathcal{S}_c(b_2b_3)\\&-\mathcal{S}(b_1)\mathcal{S}(b_2)\mathcal{S}(b_3).\end{aligned}$$

Thus, we have, with obvious notation,

$$\mathcal{S} = \sum_{m=1}^{i+j+i'+j'} \sum_{l's} \sum_{\mathrm{gr}} \mathcal{S}_c(\cdots)\mathcal{S}_c(\cdots)\cdots\mathcal{S}_c(\cdots). \tag{2.14}$$

In the case of the equilibrium system, all but $\langle a_{\mathbf{p}} a_{\mathbf{q}}^\dagger \rangle$ and $\langle a_{\mathbf{q}}^\dagger a_{\mathbf{p}} \rangle$ vanish. From the definition of $\mathcal{S}_c$, it is not difficult to show that, for $N \geqslant 3$,

$$\mathcal{S}_c(b_{j_1} b_{j_2} \cdots b_{j_l}) = \mathcal{S}_c(:b_{j_1} b_{j_2} \cdots b_{j_l}:), \tag{2.15}$$

where ":$\cdots$:" indicates to take the normal ordering with respect to the creation and annihilation operators.

C. Reaction-probability formula

Now, $\mathcal{N}$ in Eq. (2.8) may be written as

$$\mathcal{N} = \left(\prod_{j=1}^{l} \int d^4x_j d^4x_j' F_j(x_j) F_j^*(x_j') \right) \left(\prod_{j=1}^{l'} \int d^4y_j d^4y_j' G_j^*(y_j) G_j(y_j') \right)$$
$$\times \sum_{i,j,i',j'} \left(\prod_{j=1}^{i} \int d^4\xi_j \sum_{\mathbf{p}_j} \frac{1}{\sqrt{2E_{p_j}V}} e^{-iP_j\cdot\xi_j} \right) \left(\prod_{j=1}^{i'} \int d^4\zeta_j \sum_{\mathbf{q}_j} \frac{1}{\sqrt{2E_{q_j}V}} e^{iQ_j\cdot\zeta_j} \right) \left(\prod_{l=1}^{j} \int d^4\xi_j' \sum_{\mathbf{p}_j'} \frac{1}{\sqrt{2E_{p_j'}V}} e^{iP_j'\cdot\xi_j'} \right)$$
$$\times \left(\prod_{l=1}^{j'} \int d^4\zeta_j' \sum_{\mathbf{q}_j'} \frac{1}{\sqrt{2E_{q_j'}V}} e^{-iQ_j'\cdot\zeta_j'} \right) \mathcal{S}\mathcal{W}(\{x'\},\{\xi'\};\{\zeta'\},\{y'\}:\{y\},\{\zeta\};\{\xi\},\{x\})\mathbf{S}. \tag{2.16}$$

Carrying out the integration over ξ's, ζ's, ξ''s, ζ''s and the internal spacetime vertex points, which are included in $\mathcal{W}$, we obtain, with obvious notation,

$$\mathcal{N} = \left(\prod_{j=1}^{l} \int d^4x_j d^4x_j' F_j(x_j) F_j^*(x_j') \right) \left(\prod_{j=1}^{l'} \int d^4y_j d^4y_j' G_j^*(y_j) G_j(y_j') \right) \sum_{i,j,i',j'} \left(\prod_{j=1}^{i} \sum_{\mathbf{p}_j} \frac{1}{\sqrt{2E_{p_j}V}} \right) \left(\prod_{j=1}^{i'} \sum_{\mathbf{q}_j} \frac{1}{\sqrt{2E_{q_j}V}} \right)$$
$$\times \left(\prod_{l=1}^{j} \sum_{\mathbf{p}_j'} \frac{1}{\sqrt{2E_{p_j'}V}} \right) \left(\prod_{l=1}^{j'} \sum_{\mathbf{q}_j'} \frac{1}{\sqrt{2E_{q_j'}V}} \right) \mathcal{S}\mathcal{W}(\{x'\},\{\mathbf{p}'\};\{\mathbf{q}'\},\{y'\}:\{y\},\{\mathbf{q}\};\{\mathbf{p}\},\{x\})\mathbf{S}. \tag{2.17}$$

Let us Fourier transform the wave functions $F_j(x)$, $G_j(x)$:

$$F_j(x) = \int d\mathbf{r}_j e^{-iR_j\cdot(x-X_c)} \tilde{F}_j(\mathbf{r}_j),$$
$$G_j(x) = \int d\mathbf{r}_j e^{-iR_j\cdot(x-X_c)} \tilde{G}_j(\mathbf{r}_j), \tag{2.18}$$

where $R_j^\mu = (E_j, \mathbf{r}_j)$ with $E_j = \sqrt{r_j^2 + M^2}$. In Eq. (2.18), $\mathbf{X}_c$ of $X_c^\mu = (X_{c0}, \mathbf{X}_c)$ is the space point, around which Φ's are localized, and X_{c0} is the time, around which the reaction takes place. In general, $\tilde{F}_j$ and $\tilde{G}_j$ also depend on X_c.

Substituting Eq. (2.18) into Eq. (2.17) and carrying out the integration over x_j, x_j', y_j, and y_j', we obtain

$$\mathcal{N} = \left(\prod_{j=1}^{l} \int d\mathbf{r}_j d\mathbf{r}_j' \tilde{F}_j(\mathbf{r}_j) \tilde{F}_j^*(\mathbf{r}_j') \right) \left(\prod_{j=1}^{l'} \int d\mathbf{s}_j d\mathbf{s}_j' \tilde{G}_j^*(\mathbf{s}_j) \tilde{G}_j(\mathbf{s}_j') \right) \sum_{i,j,i',j'} \left(\prod_{j=1}^{i} \sum_{\mathbf{p}_j} \frac{1}{\sqrt{2E_{p_j}V}} \right) \left(\prod_{j=1}^{i'} \sum_{\mathbf{q}_j} \frac{1}{\sqrt{2E_{q_j}V}} \right)$$
$$\times \left(\prod_{l=1}^{j} \sum_{\mathbf{p}_j'} \frac{1}{\sqrt{2E_{p_j'}V}} \right) \left(\prod_{l=1}^{j'} \sum_{\mathbf{q}_j'} \frac{1}{\sqrt{2E_{q_j'}V}} \right) 2\pi\delta\left(\sum^{l} r_{j0} - \sum^{l'} s_{j0} + \sum^{i} p_0 - \sum^{i'} q_0 \right)$$
$$\times 2\pi\delta\left(\sum^{l'} s_{j0}' - \sum^{l} r_{j0}' + \sum^{j'} q_0' - \sum^{j} p_0' \right) V\delta\left(\sum^{l} \mathbf{r}_j - \sum^{l'} \mathbf{s}_j ; \sum^{i'} \mathbf{q} - \sum^{i} \mathbf{p} \right)$$
$$\times V\delta\left(\sum^{l'} \mathbf{s}_j' - \sum^{l} \mathbf{r}_j' ; \sum^{j} \mathbf{p}' - \sum^{j'} \mathbf{q}' \right) \mathcal{S}\mathcal{W}(\{\mathbf{r}'\},\{\mathbf{p}'\};\{\mathbf{q}'\},\{\mathbf{s}'\}:\{\mathbf{s}\},\{\mathbf{q}\};\{\mathbf{p}\},\{\mathbf{r}\})$$
$$\times \exp\left[i\left(\sum^{l} (R_j - R_j') - \sum^{l'} (S_j - S_j') \right) \cdot X_c \right] \mathbf{S}. \tag{2.19}$$

Note that, when $\langle S\rangle$ ($\in W$) or $\langle S^\dagger\rangle$ consists of several disconnected parts, the corresponding (momentum-conservation) δ function above becomes product of several δ functions.

The form for $\mathcal{D}$, Eq. (2.2c), is given by Eq. (2.16) or Eq. (2.19), in which factors related to the Φ fields are deleted.

In general, $\mathcal{N}$ consists of several graphically disconnected parts. As assumed in Sec. II A, all Φ's are included in one connected part $\mathcal{N}_{\rm con}$. Other parts, which we write as $\mathcal{D}$, include only the constituent particles ϕ' of the system. Then, it is obvious that $\mathcal{N}$ takes the form $\mathcal{N}=\mathcal{N}_{\rm con}\mathcal{D}$ [cf. Eq. (2.3)]. It is also obvious that $\mathcal{D}$ is a contribution to $\mathcal{D}$ in Eqs. (2.2). Then, such a contribution does contribute to the reaction probability $\mathcal{P}$, Eq. (2.2a), as $\mathcal{N}_{\rm con}$, which has already been dealt with in a lower-order level. Thus, computation of $\mathcal{N}$'s, which consist of one connected part, is sufficient.

III. OUT-OF-EQUILIBRIUM REACTION-PROBABILITY FORMULA

A. Preliminaries

In this section, we restrict our concern to quasiuniform systems near equilibrium and nonequilibrium quasistationary systems, which we simply refer to as out-of-equilibrium systems. Such systems are characterized [5] by a weak dependence of the reaction probabilities on X_c [cf. above after Eq. (2.18)]. More precisely, there exists a spacetime scale L^μ, such that the reaction probabilities do not appreciably depend on X_c, when X_c is in the spacetime region $|X_c^\mu - X_{c0}^\mu|$ $(=|\Delta X_c^\mu|)\lesssim L^\mu$ with X_{c0}^μ an arbitrary spacetime point. For such systems, the reactions are regarded as taking place in the region $|X_c^\mu - X_{c0}^\mu|\lesssim L^\mu$. Going to momentum space, this means that the contribution (to the reaction probability $\mathcal{N}$) from the state that includes a ''very soft'' momentum $|P^\mu|\lesssim 1/L^\mu$ should be small. More precisely, the contribution from the summation region in Eq. (2.16), in which at least one momentum (out of $\{\mathbf{p}_j,\mathbf{q}_j,\mathbf{p}_j',\mathbf{q}_j'\}$) is ''very soft,'' is negligibly small.[2]

[2]This is the case for most practical cases, which can be seen as follows. Let $\mathcal{T}$ be a typical scale(s) of the system under consideration. In the case of thermal-equilibrium system, $\mathcal{T}$ is the temperature of the system. As a result of interactions, an effective mass is induced and the vacuum-theory mass m turns out to the effective mass $M_{eff}(X_c)$. In the case of $m\gg\sqrt{\lambda}\mathcal{T}$, $M_{eff}(X_c)$ is not much different from m and, for $m\lesssim\sqrt{\lambda}\mathcal{T}$, a tadpole diagram induces mass of $O(\sqrt{\lambda}\mathcal{T})$, so that $M_{eff}(X_c)=O(\sqrt{\lambda}\mathcal{T})$. [$\sqrt{\lambda}\mathcal{T}$ (or even $\lambda\mathcal{T}$) is the scale that characterizes reactions. We assume that this scale is much larger than the ''very soft'' momentum scale, $1/L^\mu\ll\sqrt{\lambda}\mathcal{T}$ (or $\lambda\mathcal{T}$).] Most amplitudes, when computed in perturbation theory (to be deduced below), are insensitive to the region $|P^\mu|\leqslant O(\sqrt{\lambda}\mathcal{T})$. Then, the contribution from the region $|P^\mu|\lesssim 1/L^\mu$ is small, since the phase-space volume is small. Incidentally, in the case of equilibrium thermal QED or QCD ($m=0$), there are some quantities that diverge at infrared limits to leading order in hard-thermal-loop resummation schemes [8,9]. For such cases, a more elaborate analysis is required.

Let us pick out $\langle a_{\mathbf{p}}\rangle$ from $\mathcal{S}$ in Eq. (2.14), which appears in $\mathcal{N}$, Eq. (2.16), in the form

$$\sum_{\mathbf{p}}\frac{1}{\sqrt{2E_pV}}\langle a_{\mathbf{p}}\rangle e^{-iP\cdot\omega}, \tag{3.1}$$

where ω stands for ξ_j or ζ_j'. The above observation shows that the quantity (3.1) does not appreciably depend on ω^μ, when $|\omega^\mu - X_{c0}^\mu|\lesssim L^\mu$. This means that $\langle a_{\mathbf{p}}\rangle\simeq 0$ for $|p^i|\gtrsim 1/L^i$ and $p^0=E_p\gtrsim 1/L^0$. Then, the argument at the end of the above paragraph shows that the contribution to $\mathcal{N}$ that includes $\langle a\rangle$ can be ignored. The same reasoning shows that the contribution including $\langle a^\dagger\rangle$ and/or $\mathcal{S}_c(aa\cdots a)$ and/or $\mathcal{S}_c(a^\dagger a^\dagger\cdots a^\dagger)$ may also be ignored.

Recalling the identity (2.15), we pick out from Eq. (2.14) one $\mathcal{S}_c(a^\dagger_{\mathbf{p}_1}\cdots a^\dagger_{\mathbf{p}_j}a_{\mathbf{p}_{j+1}}\cdots a_{\mathbf{p}_n})$ $(n\geqslant 3)$. In $\mathcal{N}$ in Eq. (2.16), this factor appears in the form

$$\sum_{\{\mathbf{p}\}}\left(\prod_{l=1}^{n}\frac{1}{\sqrt{2E_{p_l}V}}\right)\mathcal{S}_c\left(\left(\prod_{l=1}^{j}a^\dagger_{\mathbf{p}_l}\right)\left(\prod_{l'=j+1}^{n}a_{\mathbf{p}_{l'}}\right)\right)\times\exp\left[i\left(\sum_{l=1}^{j}P_l\cdot z_l-\sum_{l'=j+1}^{n}P_{l'}\cdot z_{l'}\right)\right], \tag{3.2}$$

where $p_{l0}=E_p$ $(l=1,\dots,n)$. It is not difficult to show that among the contributions to $\mathcal{N}$, there are contributions whose counterparts of Eq. (3.2), together with Eq. (3.2), can be united into the form

$$\mathcal{C}(\{z\})\equiv i^{n-1}\mathcal{S}_c(:\phi(z_1)\cdots\phi(z_n):). \tag{3.3}$$

Here

$$\phi(z)=\sum_{\mathbf{p}}\frac{1}{\sqrt{2E_pV}}[a_{\mathbf{p}}e^{-iP\cdot z}+a^\dagger_{\mathbf{p}}e^{iP\cdot z}],$$

where $p_0=E_p$ and '': $\cdots$:'' in Eq. (3.3) indicates to take the normal ordering. As discussed at the beginning of this subsection, for the system under consideration, the function (3.2) does not change appreciably in the region $|\Delta Z^\mu|\lesssim L^\mu$ $(Z=\sum_{l=1}^n z_l/n)$. This leads to an approximate momentum conservation for the function (3.2):

$$\left|\sum_{l=1}^{j}P_l^\mu-\sum_{l'=j+1}^{n}P_{l'}^\mu\right|\lesssim 1/L^\mu. \tag{3.4}$$

This is also the case for $\mathcal{C}(\{z\})$ in Eq. (3.3). The conditions under which the initial correlations may be ignored are discussed in [10]. In the following, we ignore the initial correlations, the inclusion of which into the formula obtained below is straightforward.

After all this, in $\mathcal{S}$ in Eq. (2.16), we keep only $\langle a^\dagger a\rangle$'s:

$$\mathcal{S}=\sum_{m,n}\sum_{gr}\langle a^\dagger_{\mathbf{p}'_{l'_j}}a_{\mathbf{q}'_{k'_{j'}}}\rangle\cdots\langle a^\dagger_{\mathbf{p}'_{l'_{n+1}}}a_{\mathbf{q}'_{k'_{m+1}}}\rangle(\delta_{\mathbf{q}_{k_{i'}},\mathbf{q}'_{k'_m}}+\langle a^\dagger_{\mathbf{q}_{k_{i'}}}a_{\mathbf{q}'_{k'_m}}\rangle)\cdots(\delta_{\mathbf{q}_{k_{i-n+1}},\mathbf{q}'_{k'_1}}+\langle a^\dagger_{\mathbf{q}_{k_{i-n+1}}}a_{\mathbf{q}'_{k'_1}}\rangle)$$
$$\times\langle a^\dagger_{\mathbf{q}_{j_{i-n}}}a_{\mathbf{p}_{l_i}}\rangle\cdots\langle a^\dagger_{\mathbf{q}_{j_1}}a_{\mathbf{p}_{l_{n+1}}}\rangle\langle a^\dagger_{\mathbf{p}'_{l'_n}}a_{\mathbf{p}_{l_n}}\rangle\cdots\langle a^\dagger_{\mathbf{p}'_{l'_1}}a_{\mathbf{p}_{l_1}}\rangle, \tag{3.5}$$

where $j-n=j'-m$ and $i'-m=i-n$, which leads to $i+j'=j+i'$.

Referring to Eq. (2.16), we use the following set of symbols throughout in the sequel:

$$\mathcal{V}_\Phi=\mathcal{V}_\Phi^S\cup\mathcal{V}_\Phi^{S^\dagger},\quad \mathcal{V}_\Phi^S=\{x\}\cup\{y\},\quad \mathcal{V}_\Phi^{S^\dagger}=\{x'\}\cup\{y'\},$$

$$\mathcal{V}_e=\mathcal{V}_e^S\cup\mathcal{V}_e^{S^\dagger},\quad \mathcal{V}_e^S=\{\xi\}\cup\{\zeta\},\quad \mathcal{V}_e^{S^\dagger}=\{\xi'\}\cup\{\zeta'\},$$

and $\mathcal{V}_i=\mathcal{V}_i^S\cup\mathcal{V}_i^{S^\dagger}$ with $\mathcal{V}_i^S$ $[\mathcal{V}_i^{S^\dagger}]$ the set of internal-vertex points in $\langle S\rangle$ $[\langle S^\dagger\rangle]$ $(\in\mathcal{W})$. When the vertex point ξ_j (ξ'_j) or ζ_l (ζ'_l) coincides with one of the vertex points in $\mathcal{V}_\Phi^S$ $(\mathcal{V}_\Phi^{S^\dagger})$, we include it in $\mathcal{V}_\Phi^S$ $(\mathcal{V}_\Phi^{S^\dagger})$. At the final stage, $\mathcal{V}_\Phi$ $(\mathcal{V}_e\cup\mathcal{V}_i)$ turns out to be the set of external-vertex (internal-vertex) points of the out-of-equilibrium amplitude (3.19) representing $\mathcal{P}$.

B. Two-point function

From Eq. (2.16) with Eq. (3.5), we pick out

$$i\tilde{\Delta}(\rho,\sigma)\equiv\sum_{\mathbf{p},\mathbf{p}'}\frac{1}{\sqrt{2E_pV}\sqrt{2E_{p'}V}}\langle a^\dagger_{\mathbf{p}'}a_{\mathbf{p}}\rangle e^{-i(P\cdot\rho-P'\cdot\sigma)},$$

where $p_0=E_p$, $p'_0=E_{p'}$, and $\mathbf{p}\in\{\mathbf{p}\}\cup\{\mathbf{q}'\}$, $\mathbf{p}'\in\{\mathbf{p}'\}\cup\{\mathbf{q}\}$, $\rho\in\{\xi\}\cup\{\zeta'\}$, and $\sigma\in\{\xi'\}\cup\{\zeta\}$. Changing $\mathbf{p}$ and $\mathbf{p}'$ to

$$\mathbf{p}_+=(\mathbf{p}+\mathbf{p}')/2,\quad \mathbf{p}_-=\mathbf{p}-\mathbf{p}',$$

we get

$$i\tilde{\Delta}(\rho,\sigma)=\sum_{\mathbf{p}_+}\frac{1}{\sqrt{2E_+V}\sqrt{2E_-V}}e^{-iP_+\cdot(\rho-\sigma)}\tilde{N}(X,\mathbf{p}_+), \tag{3.6}$$

$$\tilde{N}(X;\mathbf{p}_+)=\sum_{\mathbf{p}_-}e^{-i(E_+-E_-)X_0}e^{i\mathbf{p}_-\cdot\mathbf{X}}\langle a^\dagger_{\mathbf{p}_+-\mathbf{p}_-/2}a_{\mathbf{p}_++\mathbf{p}_-/2}\rangle, \tag{3.7}$$

where $X=(\rho+\sigma)/2$, $E_\pm=E_{|\mathbf{p}_+\mp i\nabla_{\mathbf{X}}/2|}$, and $p^0_+=(E_++E_-)/2$. It is worth mentioning in passing that one can easily derive, from Eq. (3.7), $P\cdot\partial_X\tilde{N}=0$.

Now, Eq. (3.6) may be written as

$$i\tilde{\Delta}(\rho,\sigma)=\sum\!\!\!\!\!\!\int\ \mathcal{D}^4Pe^{-iP\cdot(\rho-\sigma)}\frac{p_0}{\sqrt{E_+E_-}}2\pi\theta(p_0)\times\delta\left(p_0^2-\left(\frac{E_++E_-}{2}\right)^2\right)\tilde{N}(X;\mathbf{p}), \tag{3.8}$$

where

$$\sum\!\!\!\!\!\!\int\ \mathcal{D}^4P\equiv\int\frac{dp_0}{2\pi}\sum_{\mathbf{p}}\frac{1}{V}.$$

As usual, we rewrite $p=|\mathbf{p}|$ in terms of p_0 by using $\delta_+(p_0^2-\cdots)$ in Eq. (3.8). In doing so we obtain

$$\frac{p_0}{\sqrt{E_+E_-}}\tilde{N}(X;\mathbf{p})\rightarrow\left[1+\frac{(\mathbf{p}\cdot\nabla_{\mathbf{X}})^2}{4p_0^4}\right]^{-1/2}\tilde{N}(X;p_0,\hat{\mathbf{p}}),$$

where $\hat{\mathbf{p}}\equiv\mathbf{p}/p$. Carrying out the derivative expansion (expansion with respect to ∂_{X_μ}) and keeping up to the second-order $\mathbf{X}$-derivative terms, we obtain

$$i\tilde{\Delta}(\rho,\sigma)=\sum\!\!\!\!\!\!\int\ \mathcal{D}^4Pe^{-iP\cdot(\rho-\sigma)}2\pi\delta_+(P^2-m^2)N(X;p_0,\hat{\mathbf{p}}), \tag{3.9}$$

where $\delta_+(P^2-m^2)=\theta(p_0)\delta(P^2-m^2)$ and

$$N(X;p_0,\hat{\mathbf{p}})=\left[1-\frac{1}{4}\frac{\overleftarrow{\partial}}{\partial m^2}(\nabla^2_{\mathbf{X}}-(\mathbf{v}\cdot\nabla_{\mathbf{X}})^2)-\frac{(\mathbf{v}\cdot\nabla_{\mathbf{X}})^2}{8p_0^2}+\cdots\right]\tilde{N}(X;p_0,\hat{\mathbf{p}}). \tag{3.10}$$

Here $\overleftarrow{\partial}/\partial m^2$ acts on $\delta_+(P^2-m^2)$ in Eq. (3.9) and $\mathbf{v}=\mathbf{p}/p_0$.

In the case of a system in which translation invariance holds, $\langle a^\dagger_{\mathbf{p}}a_{\mathbf{q}}\rangle_c\propto\delta_{\mathbf{p},\mathbf{q}}$, Eq. (3.7) tells us that $N(p_0,\hat{\mathbf{p}})=\tilde{N}(p_0,\hat{\mathbf{p}})$ is the number density of a particle with momentum $\mathbf{p}$. This allows us to interpret $N(X;p_0,\hat{\mathbf{p}})$ as the ''bare'' number density of a quasiparticle with $\mathbf{p}$ at the spacetime point X^μ. (For more details, see [10].)

C. Construction of out-of-equilibrium propagators

So far, for simplicity of presentation, we have dealt with real-scalar-field systems. The physical meaning of the propagators to be deduced below can be determined in a more transparent manner by employing a complex-scalar-field system, which we deal with in the sequel of this section. Let $a_{\mathbf{p}}$ $(a^\dagger_{\mathbf{p}})$ be an annihilation (creation) operator for a particle of momentum $\mathbf{p}$. The antiparticle counterpart of $a_{\mathbf{p}}$ $(a^\dagger_{\mathbf{p}})$ is $b_{\mathbf{p}}$ $(b^\dagger_{\mathbf{p}})$. For simplicity, we assume that the density-matrix operator ρ commutes with charge operator Q, $[\rho,Q]=0$. Then, all but $\langle a^\dagger_{\mathbf{p}}a_{\mathbf{q}}\rangle$, $\langle b^\dagger_{\mathbf{p}}b_{\mathbf{q}}\rangle$, $\langle a^\dagger_{\mathbf{p}}b^\dagger_{\mathbf{q}}\rangle$, $\langle a_{\mathbf{p}}b_{\mathbf{q}}\rangle$ vanish. The same reasoning as at the beginning of this section shows that $\langle a^\dagger_{\mathbf{p}}b^\dagger_{\mathbf{q}}\rangle$ and $\langle a_{\mathbf{p}}b_{\mathbf{q}}\rangle$ are negligibly small. Thus, we are left with $\langle a^\dagger_{\mathbf{p}}a_{\mathbf{q}}\rangle$'s and $\langle b^\dagger_{\mathbf{p}}b_{\mathbf{q}}\rangle$'s.

(A) Let us take a Feynman diagram $\mathcal{F}$ for $\mathcal{N}$ [cf. Eq. (2.16)], and pick out from $\mathcal{F}$ a vacuum-theory propagator $i\Delta^{(0)}(z_1-z_2)=\langle 0|T\phi(z_1)\phi^\dagger(z_2)|0\rangle\in\langle S\rangle(\in\mathcal{N})$. Then, we pick up the following two diagrams for $\mathcal{N}$. The first one is the same as $\mathcal{F}$, except that $i\Delta^{(0)}(z_1-z_2)$ is replaced by

$$\sum_{\mathbf{p}}\frac{1}{\sqrt{2E_pV}}e^{-iP\cdot z_1}\sum_{\mathbf{q}}\frac{1}{\sqrt{2E_qV}}e^{iQ\cdot z_2}\langle a_{\mathbf{q}}^\dagger a_{\mathbf{p}}\rangle,$$

which is involved in Eq. (2.16). The second one is the same as $\mathcal{F}$, except that $i\Delta^{(0)}(z_1-z_2)$ is replaced by

$$\sum_{\mathbf{q}}\frac{1}{\sqrt{2E_qV}}e^{-i\bar{Q}\cdot z_2}\sum_{\mathbf{p}}\frac{1}{\sqrt{2E_pV}}e^{i\bar{P}\cdot z_1}\langle b_{-\mathbf{p}}^\dagger b_{-\mathbf{q}}\rangle,$$

with $\bar{P}\equiv(E_p,-\mathbf{p})$, etc. Adding the above two contributions to the original contribution, and Fourier transforming on z_1-z_2, we obtain, for the relevant part,

$$i\Delta_{11}\left(\frac{z_1+z_2}{2};P\right)\equiv\frac{i}{P^2-m^2+i0^+}+2\pi\delta(P^2-m^2)N\left(\frac{z_1+z_2}{2};p_0,\hat{\mathbf{p}}\right).\tag{3.11}$$

Here, N with $p_0>0$ is as in Eq. (3.10) with Eq. (3.7), while, for $p_0<0$, N takes the same form (3.10) where $\tilde{N}$ is defined, with obvious notation, as

$$\tilde{N}\left(\frac{z_1+z_2}{2};p_0,\hat{\mathbf{p}}\right)=\sum_{\mathbf{p}_-}e^{i(E_+-E_-)(z_{10}+z_{20})/2}\times e^{-i\mathbf{p}_-\cdot(\mathbf{z}_1+\mathbf{z}_2)/2}\langle b_{-\mathbf{p}+\mathbf{p}_-/2}^\dagger b_{-\mathbf{p}-\mathbf{p}_-/2}\rangle,$$

with, as before, $E_\pm=E_{|\mathbf{p}_+\mp i\nabla_{\mathbf{X}}/2|}$.

As discussed at the end of the last subsection, $N(X;p_0,\hat{\mathbf{p}})$ with $p_0>0$ is the ''bare'' number density of a quasiparticle with momentum $\mathbf{p}$ at the point X^μ. Similarly, $N(X;p_0,\hat{\mathbf{p}})$ with $p_0<0$ is the ''bare'' number density of an antiquasiparticle with momentum $-\mathbf{p}$ at X^μ.

(B) Starting from $\langle S^\dagger\rangle$ $(\in\mathcal{N})$ that includes a vacuum-theory propagator $[i\Delta^{(0)}(z_1-z_2)]^*$ and proceeding as (A) above, we obtain

$$i\Delta_{22}(X;P)\equiv[i\Delta_{11}(X;P)]^*=\frac{-i}{P^2-m^2-i0^+}+2\pi\delta(P^2-m^2)\times N\left(\frac{z_1+z_2}{2};p_0,\hat{\mathbf{p}}\right).\tag{3.12}$$

(C) Let us take a set of Feynman diagrams $\mathcal{F}_1$ and $\mathcal{F}_2$. $\mathcal{F}_1$ contains [cf. Eq. (3.5)]

$$\sum_{\mathbf{p}_j}\frac{1}{\sqrt{2E_{p_j}V}}e^{-iP_j\cdot z_1}\sum_{\mathbf{q}_k}\frac{1}{\sqrt{2E_{q_k}V}}e^{iQ_k\cdot z_2}(\delta_{\mathbf{q}_k,\mathbf{p}_j}+\langle a_{\mathbf{q}_k}^\dagger a_{\mathbf{p}_j}\rangle),\tag{3.13}$$

with $z_1\in\{\zeta'\}$ $(\in\langle S^\dagger\rangle)$ and $z_2\in\{\zeta\}$ $(\in\langle S\rangle)$. $\mathcal{F}_2$ is the same as $\mathcal{F}_1$ except that Eq. (3.13) is replaced by

$$\sum_{\mathbf{q}_k}\frac{1}{\sqrt{2E_{q_k}V}}e^{-i\bar{Q}_k\cdot z_2}\sum_{p_j}\frac{1}{\sqrt{2E_{p_j}V}}e^{i\bar{P}_j\cdot z_1}\langle b_{-\mathbf{p}_j}^\dagger b_{-\mathbf{q}_k}\rangle,$$

with $z_1\in\{\xi'\}$ $(\in\langle S^\dagger\rangle)$ and $z_2\in\{\xi\}$ $(\in\langle S\rangle)$. Adding the contributions from $\mathcal{F}_1$ and from $\mathcal{F}_2$, we extract the relevant part, of which the Fourier transformation on z_1-z_2 is

$$i\Delta_{21}(X;P)\equiv2\pi\delta(P^2-m^2)\left[\theta(p_0)+N\left(\frac{z_1+z_2}{2};p_0,\hat{\mathbf{p}}\right)\right].\tag{3.14}$$

(D) Let us take a set of Feynman diagrams $\mathcal{F}_1'$ and $\mathcal{F}_2'$. $\mathcal{F}_1'$ contains

$$\sum_{\mathbf{p}_j}\frac{1}{\sqrt{2E_{p_j}V}}e^{-iP_j\cdot z_1}\sum_{\mathbf{q}_k}\frac{1}{\sqrt{2E_{q_k}V}}e^{iQ_k\cdot z_2}\langle a_{\mathbf{q}_k}^\dagger a_{\mathbf{p}_j}\rangle,\tag{3.15}$$

with $z_1\in\{\xi\}$ $(\in\langle S\rangle)$ and $z_2\in\{\xi'\}$ $(\in\langle S^\dagger\rangle)$. $\mathcal{F}_2'$ is the same as $\mathcal{F}_1'$ except that Eq. (3.15) is replaced by

$$\sum_{\mathbf{q}_k}\frac{1}{\sqrt{2E_{q_k}V}}e^{-i\bar{Q}_k\cdot z_2}\sum_{\mathbf{p}_j}\frac{1}{\sqrt{2E_{p_j}V}}\times e^{i\bar{P}_j\cdot z_1}(\delta_{\mathbf{p}_j,\,\mathbf{q}_k}+\langle b_{-\mathbf{p}_j}^\dagger b_{-\mathbf{q}_k}\rangle),$$

with $z_1\in\{\zeta\}$ $(\in\langle S\rangle)$ and $z_2\in\{\zeta'\}$ $(\in\langle S^\dagger\rangle)$. Adding the contributions from $\mathcal{F}_1'$ and from $\mathcal{F}_2'$, we extract the relevant part, of which the Fourier transformation on z_1-z_2 is

$$i\Delta_{12}(X;P)\equiv2\pi\delta(P^2-m^2)\left[\theta(-p_0)+N\left(\frac{z_1+z_2}{2};p_0,\hat{\mathbf{p}}\right)\right].\tag{3.16}$$

The above derivation of $i\Delta_{ij}$ $(i,j=1,2)$ is self-explanatory for their physical meaning or interpretation. The physical interpretation is summarized as generalized cutting rules, which is a generalization of Cutkosky's cutting rules in vacuum theory. (For more details, see [4].)

D. Closed-time-path formalism

$i\Delta_{ij}$ $(i,j=1,2)$ obtained above are nothing but the propagators in the closed-time path (CTP) formalism of out-of-equilibrium quantum field theory [5]. The CTP formalism is constructed on the directed time path $C=C_1\oplus C_2$ in a complex-time plane, where $C_1=(-\infty\to+\infty)$ and $C_2=(+\infty\to-\infty)$. A field $\phi(x_0,\mathbf{x})$ with $x_0\in C_1$ $[x_0\in C_2]$ is denoted by $\phi_1(x_0,\mathbf{x})$ $[\phi_2(x_0,\mathbf{x})]$ and is called a type-1 [type-2] field. The interaction Lagrangian density is of the form

$$\mathcal{L}_{\rm int}=\mathcal{L}_{\rm int}^{(1)}-\mathcal{L}_{\rm int}^{(2)},$$

$$\mathcal{L}_{\rm int}^{(i)}=-\frac{\lambda}{4}(\phi_i^\dagger\phi_i)^2-\frac{g}{(n!)^2}\Phi_i(\phi_i^\dagger\phi_i)^n \quad (i=1,2).$$

Then, the vertex factor for the ''type-1 vertex'' that comes from $\mathcal{L}_{int}^{(1)}$ is the same as in vacuum theory, while the vertex factor for the ''type-2 vertex'' is minus the corresponding ''type-1 vertex factor.'' The CTP propagators are defined by the statistical average of the time-path-ordered product of fields, which are written as

$$i\Delta_{11}(x,y)=\langle \mathrm{T}\phi_1(x)\phi_1^\dagger(y)\rangle_c,$$

$$i\Delta_{22}(x,y)=\langle \overline{\mathrm{T}}\phi_2(x)\phi_2^\dagger(y)\rangle_c$$
$$=[i\Delta_{11}(y,x)]^*,$$

$$i\Delta_{12}(x,y)=\langle \phi_2^\dagger(y)\phi_1(x)\rangle_c,$$

$$i\Delta_{21}(x,y)=\langle \phi_2(x)\phi_1^\dagger(y)\rangle_c, \tag{3.17}$$

where T $(\overline{T})$ is the time-ordering (anti-time-ordering) symbol. In computing Eq. (3.17), one identifies ϕ_2 with ϕ_1. Comparing Eq. (3.17) with the above deduction of Δ_{ij} $(i,j=1,2)$, Eqs. (3.11), (3.12), (3.14), and (3.16), we see that x of $\phi_1(x)$ in Eq. (3.17) corresponds to a vertex point in $\langle S\rangle$ $(\in\mathcal{W})$ and x of $\phi_2(x)$ corresponds to a vertex point in $\langle S^\dagger\rangle$. The vertex factors in $\langle S\rangle$ $(\in\mathcal{W})$ are $-i\lambda$ for the $-\lambda(\phi^\dagger\phi)^2/4$ interaction and $-ig$ for the $-g\Phi(\phi^\dagger\phi)^n/(n!)^2$ interaction. Then, the vertex factors in $\langle S^\dagger\rangle$ $(\in\mathcal{W})$ are, in corresponding order to the above, $i\lambda$ and ig. This is in accordance with the above-mentioned vertex factors in the CTP formalism.

E. Reaction-probability formula

The observation made so far shows that $\mathcal{N}$ in Eq. (2.16) with Eq. (3.5) corresponds to an amplitude in the CTP formalism of the ''process''

$$\sum_{j=1}^{l}\Phi_{1j}+\sum_{j=1}^{l'}\Phi_{2j}\to\sum_{j=1}^{l}\Phi_{2j}+\sum_{j=1}^{l'}\Phi_{1j}. \tag{3.18}$$

As mentioned at the end of Sec. II, only connected $\mathcal{N}$'s contribute to the reaction probability $\mathcal{P}$. Thus, we finally obtain

$$\mathcal{P}=\left(\prod_{j=1}^{l}\int d^4x_jd^4x_j'F_j(x_j)F^*(x_j')\right)$$
$$\times\left(\prod_{j=1}^{l'}\int d^4y_jd^4y_j'G_j^*(y_j)G_j(y_j')\right)$$
$$\times\sum_{\rm diagrams}\int d^4\omega_1\cdots\omega_{N_d}\mathcal{F}_i(X;\{(\omega_k-\omega_{k'})\}), \tag{3.19}$$

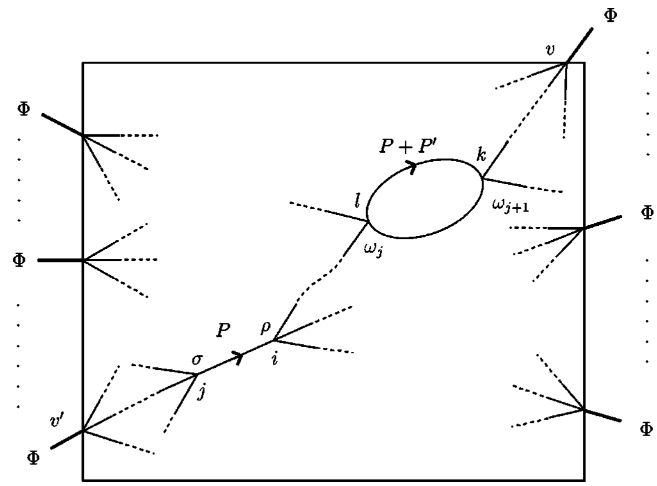

FIG. 1. A diagram for $\mathcal{F}_i$ in Eq. (3.19). i, j, k, and l are the vertex type. Each Φ is either type 1 or type 2.

where $\mathcal{F}_i$ is a *connected* amplitude in the CTP formalism which includes all Φ's. In Eq. (3.19), we have used $\{\omega\}$ for collectively denoting all the (external and internal) vertex points and the summation runs over diagrams. A pair of ω's, ω_k, and $\omega_{k'}$, in a pair of brackets $(\cdots)$ in $\mathcal{F}_i$ denotes the vertex points that are connected by $i\Delta_{kl}(\omega_{k(k')},\omega_{k'(k)})$.

Here some remarks are in order.

(1) As mentioned at the beginning of this section, inclusion of the initial correlations (3.2) or (3.3) is straightforward.

(2) Taking the infinite-volume limit $V\to\infty$ goes as follows:

$$\sum_{\mathbf{p}}\to\frac{V}{(2\pi)^3}\int d^3p,$$

$$a_{\mathbf{p}}\to\sqrt{\frac{(2\pi)^3}{V}}a(\mathbf{p}), \quad \text{etc.}$$

The above deduction shows that there is no finite-volume correction, in the sense that there do not exist extra contributions to $\mathcal{N}$, which disappear in the limit $V\to\infty$. It should be stressed that this statement holds for periodic boundary conditions.

(3) It is clear from the above deduction (cf. Secs. III B and III C) that the CTP formalism here is formulated in terms of the ''bare'' number density of quasiparticles. A canonical CTP formalism is formulated in terms of the physical or observed number density of quasiparticles. How to translate the former into the latter is discussed in [10].

Finally, we make a comment on gauge theories. If we choose a physical gauge like the Coulomb gauge or the Landshoff-Rebhan variant [11] of a covariant gauge, the gauge boson may be dealt with in a similar manner to the above scalar-field case. If we adopt a traditional covariant gauge, a straightforward modification is necessary.

IV. COMPUTATIONAL PROCEDURE

In this section, we present a concrete procedure of computing the reaction probability $\mathcal{P}$ up to nth-order terms with respect to the X_μ derivatives.

(1) From $\mathcal{F}_i$ in Eq. (3.19), we pick out $i\Delta_{ij}(\rho,\sigma)$,

$$\Delta_{ij}(\rho,\sigma)=\sum\!\!\!\!\!\!\int \mathcal{D}^4P e^{-iP\cdot(\rho-\sigma)}\Delta_{ij}\left(\frac{\rho+\sigma}{2};P\right). \quad (4.1)$$

Since $\mathcal{F}_i$ includes Φ's, the vertex point ρ $[\sigma]$ is connected[3] with a vertex point $v\,[v']\in\mathcal{V}_\Phi$ (cf. Fig. 1):

$$\frac{\rho+\sigma}{2}=\frac{1}{2}\left[-\sum_{j=0}^{k}(\omega_{j+1}-\omega_j)+\sum_{j=0}^{k'}(\omega'_j-\omega'_{j+1})+v+v'\right], \quad (4.2)$$

where $\omega_0=\rho$, $\omega'_0=\sigma$, $\omega_{k+1}=v$, $\omega'_{k'+1}=v'$, with $v,v'\in\mathcal{V}_\Phi$. In Eq. (4.2), each pair of spacetime points in a pair of brackets, ω_{j+1} and ω_j $[\omega'_j$ and $\omega'_{j+1}]$, is connected by one or several $i\Delta_{kl}(\omega_{j+1},\omega_j)$ $[i\Delta_{k'l'}(\omega'_j,\omega'_{j+1})]$ in $\mathcal{F}_i$ (cf. Fig. 1). Here, we note that v and v' may be written as

$$v=X+\tilde{v}, \quad v'=X+\tilde{v}', \quad (4.3)$$

where X is the midpoint of the external-vertex points, around which the reaction is taking place:

$$X=\frac{1}{2(l+l')}\left[\sum_{j=1}^{l}(x_j+x'_j)+\sum_{j=1}^{l'}(y_j+y'_j)\right].$$

(2) Using Eqs. (4.2) and (4.3), we expand $\Delta_{ij}((\rho+\sigma)/2;P)$ in Eq. (4.1) as

$$\Delta_{ij}\left(\frac{\rho+\sigma}{2};P\right)=\Delta_{ij}(X;P)+\frac{1}{2}\left[-\sum_{j=0}^{k}(\omega_{j+1}-\omega_j)+\sum_{j=0}^{k'}(\omega'_j-\omega'_{j+1})+\tilde{v}+\tilde{v}'\right]\cdot\partial_X\Delta_{ij}(X;P)+\cdots, \quad (4.4)$$

where "$\cdots$" stands for terms with higher-order derivatives with respect to X. The series (4.4) is truncated at the nth-order terms with respect to the X_μ derivatives. The approximation in which "$\cdots$" is ignored is called the gradient approximation.

(3) Let us deal with the term with $(\omega_{j+1}-\omega_j)$ in Eq. (4.4). It can easily be shown that $(\omega_{j+1}-\omega_j)i\Delta_{kl}(\omega_{j+1},\omega_j)$ becomes[4]

$$(\omega_{j+1}-\omega_j)^\mu\sum\!\!\!\!\!\!\int \mathcal{D}^4P' e^{-i(P+P')\cdot(\omega_{j+1}-\omega_j)}\times i\Delta_{kl}\left(\frac{\omega_{j+1}+\omega_j}{2};P+P'\right)$$
$$=\sum\!\!\!\!\!\!\int \mathcal{D}^4P' e^{-i(P+P')\cdot(\omega_{j+1}-\omega_j)}\times\frac{\partial}{i\partial P'_\mu}i\Delta_{kl}\left(\frac{\omega_j+\omega_{j+1}}{2};P+P'\right).$$

Other terms and higher X^μ-derivative terms "$\cdots$" in Eq. (4.4) may be dealt with similarly. All other parts of $\mathcal{F}_i$, Eq. (3.19), than the one (4.1) may be dealt with similarly.

(4) Carrying out the integrations over all vertex points except those in $\mathcal{V}_\Phi$, we have momentum-conservation δ functions at each internal vertex point.

As discussed at the beginning of Sec. III, the wave functions of Φ's should be localized within the space region $\lesssim L^i$ $(i=1,2,3)$. However, for simplicity, we assume in the sequel that the wave functions of Φ's are of plane-wave form[5]

$$F_j(x)=e^{-iR_j\cdot x}/(2V\sqrt{r_j^2+M^2})^{1/2},$$
$$G_j(y)=e^{-iR'_j\cdot y}/(2V\sqrt{r_j'^2+M^2})^{1/2}. \quad (4.5)$$

(5) We carry out the integrations over all vertex points in $\mathcal{V}_\Phi$ to yield momentum-conservation δ functions at those vertex points and we are left with integrations over the independent or loop momenta. Keeping the terms up to the nth-order terms with respect to the X_μ derivatives, we obtain the final formula, which may be written in the form

$$\mathcal{P}=\int d^4X\,A(X;R'_1,\ldots,R'_{l'};R_1,\ldots,R_l). \quad (4.6)$$

Note that A depends weakly on X through $N(X;Q_k)$'s. From Eq. (4.6), we see that A is the reaction rate per unit volume. Incidentally, were it not for this X dependence, integration

---

[3] Note that, in general, the vertex points v and v' are not uniquely singled out. (v can coincide with v'.) However, different choices of v and v' lead to the same reaction probability $\mathcal{P}$ within the accuracy under consideration.

[4] As in the case of some self-energy-type subdiagram, there are several $i\Delta_{kl}(\omega_{j+1},\omega_j)$'s $[i\Delta_{k'l'}(\omega'_j,\omega'_{j+1})$'s] (cf. Fig. 1). In such a case, one chooses any one of them.

[5] It is to be noted that, if we use the plane-wave form (4.5) in Eq. (2.19), the X dependence disappears. In the procedure presented here, the X dependence of $\mathcal{N}$ is already (partially) taken into account before arriving at (4).

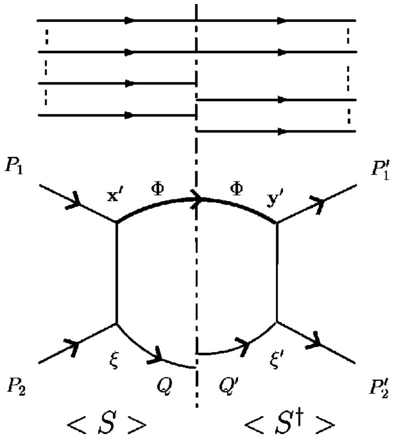

FIG. 2. A diagram representing $\mathcal{N}$, Eq. (2.2b), for the process (4.7). The spacetime points ξ and x' (ξ' and y') are connected by a vacuum-theory propagator. The dot-dashed line stands for the final-state-cut line. The group of particles on top of the figure represents the spectator particles.

over X in Eq. (4.6) would yield VT, where V is the volume of the system and $T=t_f-t_i$ is the time interval during which the reaction takes place. In the limit $V,T\to\infty$, the VT becomes

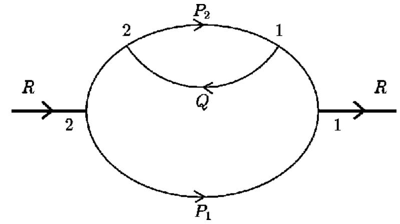

FIG. 3. An amplitude for the ''process'' $\Phi_2(R)\to\Phi_1(R)$ in the CTP formalism, a part of which represents the contribution (4.8).

$$\lim_{V,T\to\infty} VT=(2\pi)^4\delta^4(0).$$

Example

Here, for the purpose of illustration, we deal with the heavy-Φ production process

$$\text{out-of-equilibrium system }\to\Phi+\text{anything}. \tag{4.7}$$

The system is composed of a real scalar ϕ with $\mathcal{L}_{\rm int}=-\lambda\phi^3/3!$, and Φ interacts with ϕ through $\mathcal{L}_{\phi\Phi}=-g\Phi\phi^2/2$. We analyze the contribution from Fig. 2 for $\mathcal{P}$ in Eqs. (2.2). Using Eq. (2.16), we have

$$\mathcal{N}=g^2\lambda^2\int d^4x'G^*(x')\int d^4y'G(y')\sum_{\mathbf{p}_1}\frac{1}{\sqrt{2E_{p_1}V}}e^{-iP_1\cdot x'}\int d^4\xi\sum_{\mathbf{p}_2}\frac{1}{\sqrt{2E_{p_2}V}}e^{-iP_2\cdot\xi}$$
$$\times\sum_{\mathbf{q}}\frac{1}{\sqrt{2E_qV}}e^{iQ\cdot\xi}\sum_{\mathbf{p}_1'}\frac{1}{\sqrt{2E_{p_1'}V}}e^{iP_1'\cdot y'}\int d^4\xi'\sum_{\mathbf{p}_2'}\frac{1}{\sqrt{2E_{p_2'}V}}e^{iP_2'\cdot\xi'}$$
$$\times\sum_{\mathbf{q}'}\frac{1}{\sqrt{2E_{q'}V}}e^{-iQ'\cdot\xi'}\mathcal{S}i\Delta^{(0)}(\xi-x')(i\Delta^{(0)}(\xi'-y'))^*,$$

where $\Delta^{(0)}$ is the vacuum-theory propagator of ϕ and $\mathcal{S}$ [cf. Eq. (2.10)] takes the form

$$\mathcal{S}=\langle a^\dagger_{\mathbf{p}_1'}a^\dagger_{\mathbf{p}_2'}a_{\mathbf{q}'}a^\dagger_{\mathbf{q}}a_{\mathbf{p}_1}a_{\mathbf{p}_2}\rangle=\langle a^\dagger_{\mathbf{p}_1'}a^\dagger_{\mathbf{p}_2'}(\delta_{\mathbf{q}',\mathbf{q}}+a^\dagger_{\mathbf{q}}a_{\mathbf{q}'})a_{\mathbf{p}_1}a_{\mathbf{p}_2}\rangle.$$

We compute the contributions that include only two-point functions. If necessary, the contributions including initial correlations may be written down in a straightforward manner. Keeping the terms that do not vanish kinematically, we have

$$\mathcal{S}=\mathcal{S}_1+\mathcal{S}_2,\quad \mathcal{S}_1=\langle a^\dagger_{\mathbf{p}_1'}a_{\mathbf{p}_1}\rangle\langle a^\dagger_{\mathbf{p}_2'}a_{\mathbf{p}_2}\rangle[\delta_{\mathbf{q},\mathbf{q}'}+\langle a^\dagger_{\mathbf{q}}a_{\mathbf{q}'}\rangle],\quad \mathcal{S}_2=\mathcal{S}_1|_{\mathbf{p}_1\leftrightarrow\mathbf{p}_2}.$$

We compute the contribution $\mathcal{N}_1$ from $\mathcal{S}_1$. The contribution from $\mathcal{S}_2$ may be computed similarly. Following the procedure presented above, we obtain

$$\mathcal{N}_1=g^2\lambda^2\int d^4x'G^*(x')\int d^4y'G(y')\int d^4\xi\int d^4\xi'\sum\!\!\!\!\!\!\int\ \mathcal{D}^4P_1e^{-iP_1\cdot(x'-y')}2\pi\delta_+(P_1^2-m^2)N\left(\frac{x'+y'}{2};P_1\right)$$
$$\times\sum\!\!\!\!\!\!\int\ \mathcal{D}^4P_2e^{-iP_2\cdot(\xi-\xi')}2\pi\delta_+(P_2^2-m^2)N\left(\frac{\xi+\xi'}{2};P_2\right)\sum\!\!\!\!\!\!\int\ \mathcal{D}^4Qe^{-iQ\cdot(\xi'-\xi)}2\pi\delta_+(Q^2-m^2)\left\{1+N\left(\frac{\xi+\xi'}{2};Q\right)\right\}$$
$$\times\sum\!\!\!\!\!\!\int\ \mathcal{D}^4P'e^{-iP'\cdot(\xi-x')}\frac{i}{P'^2-m^2+i0^+}\sum\!\!\!\!\!\!\int\ \mathcal{D}^4Q'e^{-iQ'\cdot(y'-\xi')}\frac{-i}{Q'^2-m^2-i0^+}.$$

Here we observe that

$$\frac{\xi+\xi'}{2}-\frac{x'+y'}{2}=\frac{1}{2}[(\xi-x')+(\xi'-y')]\to\frac{-i}{2}\left(\frac{\partial}{\partial P'}-\frac{\partial}{\partial Q'}\right),$$

where the partial derivatives are understood to act on the ''propagators'' in momentum representation.

Making the plane-wave approximation for $G(x)$,

$$G(x)=\frac{e^{-iR\cdot x}}{\sqrt{2E_\Phi V}}\quad (E_\Phi=\sqrt{r^2+M^2}),$$

we finally obtain, within the gradient approximation,

$$\begin{aligned}\mathcal{N}_1\simeq\frac{g^2\lambda^2}{2E_\Phi V}\int d^4X \sum\!\!\!\!\!\!\int \mathcal{D}^4P_1 \sum\!\!\!\!\!\!\int \mathcal{D}^4P_2[2\pi\delta_+(P_1^2-m^2)\tilde{N}(X;P_1)][2\pi\delta_+(P_2^2-m^2)\tilde{N}(X_2;P_2)]\\ \times[2\pi\delta_+(Q^2-m^2)\{1+\tilde{N}(X_1;Q)\}]\left[1-\frac{i}{2}(\overleftarrow{\partial}_{X_2}+\overleftarrow{\partial}_{X_1})\cdot\left(\frac{\vec{\partial}}{\partial P'}-\frac{\vec{\partial}}{\partial Q'}\right)\right]\frac{1}{P'^2-m^2+i0^+}\\ \times\frac{1}{Q'^2-m^2-i0^+}\Bigg|_{X_1=X_2=X,Q'=P'},\end{aligned}\qquad(4.8)$$

where $X=(x'+y')/2$ and $P'=Q'=P_1-R$ and $Q=P_2+P_1-R$.

Equation (4.8) corresponds to a contribution to the amplitude in the CTP formalism of the ''process'' [cf. Eq. (3.18)], $\Phi_2(R)\to\Phi_1(R)$, and constitutes a part of the diagram as depicted in Fig. 3 in the CTP formalism. As a matter of fact, Eq. (4.8) represents Fig. 3 with $(p_{10}>0,p_{20}>0,q_0>0)$ plus Fig. 3 with $(p_{10}>0,p_{20}<0,q_0<0)$.

[1] H. A. Weldon, Phys. Rev. D **28**, 2007 (1983).

[2] A. Niégawa, Phys. Lett. B **247**, 351 (1990); N. Ashida, H. Nakkagawa, A. Niégawa, and H. Yokota, Ann. Phys. (N.Y.) **215**, 315 (1992); **230**, 161(E) (1994); Phys. Rev. D **45**, 2066 (1992); N. Asida, Int. J. Mod. Phys. A **8**, 1729 (1993); **8**, 3829(E) (1993); A. Niégawa and K. Takashiba, Ann. Phys. (N.Y.) **226**, 293 (1993); **230**, 162(E) (1994).

[3] M. Jacob and P. V. Landshoff, Phys. Lett. B **281**, 114 (1992); P. V. Landshoff, *ibid.* **386**, 291 (1996).

[4] A. Niégawa, Phys. Rev. D **57**, 1379 (1998).

[5] K.-C. Chou, Z.-B. Su, B.-L. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985).

[6] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[7] A. J. Niemi and G. W. Semenoff, Ann. Phys. (N.Y.) **152**, 105 (1984).

[8] R. D. Pisarski, Phys. Rev. Lett. **63**, 1129 (1989); E. Braaten and R. D. Pisarski, Nucl. Phys. **B337**, 569 (1990); **B339**, 310 (1990); J. Frenkel and J. C. Taylor, *ibid.* **B334**, 199 (1990). For an out-of-equilibrium generalization, see, e.g., M. E. Carrington, H. Defu, and M. H. Thoma, Eur. Phys. J. C **7**, 347 (1999).

[9] M. Le Bellac, *Thermal Field Theory* (Cambridge University Press, Cambridge, England, 1996).

[10] A. Niégawa, Prog. Theor. Phys. **102**, 1 (1999).

[11] P. V. Landshoff and A. Rebhan, Nucl. Phys. **B383**, 607 (1992).